$\Rightarrow \int p\,dp + \int q\,dq = 0$

$\Rightarrow \frac{1}{2}(p^2+q^2) = \frac{a^2}{2}$

$\Rightarrow p^2+q^2 = a^2$

$\therefore$ From (i) we get..

$a^2x = pz$

$\Rightarrow p = \frac{a^2x}{z}$

$\because q = \sqrt{a^2-p^2} = \sqrt{a^2 - \frac{a^4x^2}{z^2}}$

$= \frac{a}{z}\sqrt{z^2-a^2x^2}$

Now from $dz = p\,dx + q\,dy$ we get..

$dz = \frac{a^2x}{z}dx + \frac{a}{z}\sqrt{z^2-a^2x^2}\,dy$

$z\,dz - a^2x\,dx = a\sqrt{z^2-a^2x^2}\,dy$

$\frac{1}{2}\frac{d(z^2-a^2x^2)}{\sqrt{z^2-a^2x^2}} = a\,dy$

$\Rightarrow \int \frac{d(z^2-a^2x^2)}{\sqrt{z^2-a^2x^2}} = \int 2a\,dy$

$\Rightarrow 2\sqrt{z^2-a^2x^2} = 2ay + 2b$

$\Rightarrow \sqrt{z^2-a^2x^2} = ay+b$

$\Rightarrow z^2-a^2x^2 = (ay+b)^2$

$\Rightarrow z^2 = a^2x^2 + (ay+b)^2$

• ~~Some~~ Standard type of PDE :-

• Type - I :- $f(p,q) = 0$

Charpit's A.E. are

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{dp}{0} = \frac{dq}{0}$$

$\therefore dp = 0 \Rightarrow p = a$ (constant)

$\therefore q = Q(a)$ (constant)

$\therefore dz = p\,dx + q\,dy$

$\Rightarrow dz = a\,dx + Q(a)\,dy$

$\Rightarrow z = ax + Q(a)y + b$

e.g. (i) $p + q = pq$

$f(p,q) = p + q - pq = 0$

$\therefore p = a$ (constant)

$\therefore q = \frac{a}{a-1} = Q(a)$

$\therefore dz = p\,dx + q\,dy$

$\Rightarrow dz = a\,dx + \frac{a}{a-1}\,dy$

$\Rightarrow z = ax + \frac{a}{a-1}y + b$

(ii) $p^2 + q^2 = 1$

$\therefore f(p,q) = p^2 + q^2 - 1 = 0$

$\therefore p = a$ (constant)

$q = \sqrt{1-a^2} = Q(a)$

$\therefore z = ax + \sqrt{1-a^2}\,y + b$

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• Type - II :-

$f(z, p, q) = 0$

Charpit's A.E are..

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{dp}{-pf_z} = \frac{dq}{-qf_z}$$

$$\therefore \frac{dp}{p} = \frac{dq}{q}$$

$$\Rightarrow p = aq$$

$$\therefore q = Q(a, z)$$

$$\therefore dz = p\,dx + q\,dy$$

$$\Rightarrow dz = aQ\,dx + Q\,dy$$

$$\Rightarrow \int \frac{dz}{Q(a,z)} = ax + y$$

e.g.: $zpq = p + q$

$$\Rightarrow f(z, p, q) = zpq - (p+q) = 0$$

Charpit's A.E. are

$$\frac{dx}{zq - 1} = \frac{dy}{zp - 1} \quad \frac{dz}{p(zq-1) + q(zp-1)} = \frac{dp}{-p\ pq} = \frac{dq}{-q \cdot pq}$$

$$\therefore \frac{dp}{p} = \frac{dq}{q}$$

$$\Rightarrow p = aq$$

$$\therefore z \cdot aq^2 = aq + q \quad \Rightarrow q = \frac{1+a}{az}$$

$$\therefore p = \frac{1+a}{z}$$

$$\therefore dz = p\,dx + q\,dy$$

$$\Rightarrow dz = \frac{1+a}{z}dx + \frac{1+a}{az}dy$$

$$\Rightarrow \int z\,dz = (1+a)\int d\left(x + y/a\right)$$

$$\Rightarrow \frac{z^2}{2} = \frac{a+1}{a}(ax + y) + b$$

- **Type - III :-**

$$f(x, p) = g(y, q)$$

$$p^2 + q^2 = x + y$$

$\Rightarrow \quad p^2 - x = y - q^2 = a \text{ (say)}$

$$p = \sqrt{a+x}$$

$$q = \sqrt{y-a}$$

$\therefore \quad dz = \sqrt{a+x}\, dx + \sqrt{y-a}\, dy$

$\Rightarrow \quad z = \frac{2}{3}(a+x)^{3/2} + \frac{2}{3}(y-a)^{3/2} + b$

- **Type - IV :-**

$$z = px + qy + f(p, q) \rightarrow z = ax + by + f(a, b)$$

Charpit's A.E are.

$$\frac{dx}{x + f_p} = \frac{dy}{y + f_q} = \frac{dz}{p(x + f_p) + q(y + f_q)} = \frac{dp}{p - p} = \frac{dq}{q - q}$$

$\therefore \quad dp = 0 \Rightarrow p = a$

$\quad\quad dq = 0 \Rightarrow q = b$

$\therefore \quad z = ax + by + f(a, b)$

e.g. $\quad z = px + qy + \sqrt{1 + p^2 + q^2}$

$\Rightarrow$ Charpits A.E. are...

$$\frac{dx}{x + \frac{\partial}{\partial p}(\sqrt{1+p^2+q^2})} = \frac{dy}{y + \frac{\partial}{\partial q}(\sqrt{1+p^2+q^2})} = \frac{dz}{p\left\{x + \frac{\partial}{\partial p}\sqrt{1+p^2+q^2}\right\} + q\left\{y + \frac{\partial}{\partial q}\sqrt{1+p^2+q^2}\right\}} = \frac{dp}{p - p} = \frac{dq}{q - q}$$

$\therefore \quad dp = 0 \Rightarrow p = a$

$\quad\quad dq = 0 \Rightarrow q = b$

$\therefore \quad z = ax + by + \sqrt{1 + a^2 + b^2}$

- **Linear PDE with constant co-efficients :-**

Let $(A_0 D^n + A_1 D^{n-1} D' + A_2 D^{n-2} D'^2 + \dots + A_n D'^n) z = f(x, y)$

be the linear PDE with $A_0, A_1 \dots A_n$ are constants and which can be written as $F(D, D')z = f(x, y)$ where $D = \frac{\partial}{\partial x}$, $D' = \frac{\partial}{\partial y}$

- Theorems (Statement) :-

I :- If u is the complimentary function and z' be a particular integral of a linear PDE, $F(D, D')z = f(x, y)$ then $u + z'$ is a general solution of the PDE.

II :- If $u_1, u_2 \dots u_n$ are solutions of the homogeneous linear PDE, $F(D, D')z = 0$, $\sum_{r=1}^{n} c_r u_r$ is also a solution of the PDE. $c_1, c_2 \text{ — } c_n$ are arbitary constants.

III :- $\alpha_r D + \beta_r D' + \gamma_r$ is a factor of $F(D, D') = 0$. $\phi(\xi)$ is a arbitary function of single variable ξ, then if $\alpha_r \neq 0$, $u_r = \exp\left(-\frac{\gamma_r}{\alpha_r} x\right) \phi_r(\beta_r x - \alpha_r y)$ is a solution of $F(D, D')z = 0$

IV :- If $\beta_r D' + \gamma_r$ is a factor of $F(D, D')$ and if $\phi_r(\xi)$ is a arbitary function of ξ, then if $\beta_r \neq 0$, $u_r = \exp\left(-\frac{\gamma_r}{\beta_r} y\right) \phi(\beta_r x)$ is a solution of the PDE, $F(D, D')z = 0$

V :- If $(\alpha_r D + \beta_r D' + \gamma_r)^n$, $\alpha_r \neq 0$ is a factor of $F(D, D')$ and if the functions $\phi_{r1}, \phi_{r2} \dots \phi_{rn}$ are arbitary, then $\exp\left(-\frac{\gamma_r x}{\alpha_r}\right) \sum_{s=1}^{n} x^{s-1} \phi_{rs}(\beta_r x - \alpha_r y)$ is a solution of $F(D, D')z = 0$

VI :- If $(\beta_r D' + \gamma_r)^n$, $\beta_r \neq 0$ is a factor of $F(D, D')$ and if the functions $\phi_{r1}, \phi_{r2} \dots \phi_{rn}$ are arbitary, then $\exp\left(-\frac{\gamma_r y}{\beta_r}\right) \sum_{s=1}^{n} x^{s-1} \phi_{rs}(\beta_r x)$ is a solution of $F(D, D')z = 0$

• $(D^3 - 6D^2D' + 11DD'^2 - 6D'^3)z = 0$

$\Rightarrow (D^3 - 6D^2D' + 11DD' - 6D'^3)z = 0$

$\Rightarrow (D-D')(D-2D')(D-3D')z = 0$

$\therefore$ The G S is ..

$$z = \phi_1(-x-y) + \phi_2(-2x-y) + \phi_3(-3x-y)$$

or $z = \phi_1(y+x) + \phi_2(y+2x) + \phi_3(y+3x)$, where ϕ_1, ϕ_2, ϕ_3 being arbitary functions.

• $(2D^2 + 5DD' + 2D'^2)z = 0$

$\Rightarrow (2D^2 + 5DD' + 2D'^2)z = 0$

$\Rightarrow (2D + D')(D + 2D')z = 0$

$\therefore$ The G. S. is ..

$$z = \phi_1(x-2y) + \phi_2(2x-y)$$

$\Rightarrow z = \phi_1(y - x/2) + \phi_2(y-2x)$, where ϕ_1, ϕ_2 being arbitary functions

• Method of finding PI :-

I :- If $F(D, D')$ be homogeneous function of D and D' of degree n, then

$$\frac{1}{F(D,D')}\phi^n(ax+by) = \frac{1}{F(a,b)}\phi^n(ax+by)$$

e.g. $(D^2 + 3DD' + 2D'^2)z = x+y$

$\Rightarrow (D^2 + 3DD' + 2D'^2)z = x+y$

$\Rightarrow (D+2D')(D+D')z = x+y$

$\therefore$ C.F $= \phi_1(2x-y) + \phi_2(y-x)$

$= \phi_1(y-2x) + \phi_2(y-x)$, where ϕ_1, ϕ_2 being arbitary functions

$\therefore$ P.I. $= \dfrac{1}{D^2 + 3DD' + 2D'^2} (x+y)$

$= \dfrac{1}{1^2 + 3 + 2.1^2} \iint v \, dv \, dv$ where $v = x+y$

$= \dfrac{1}{6} \cdot \dfrac{1}{2} \int v^2 dv$

$= \dfrac{1}{6} \dfrac{1}{2} \cdot \dfrac{1}{3} v^2 = \dfrac{1}{36} (x+y)^3$

$\therefore$ The general solution is - -

$$Z = \phi_1 (y-2x) + \phi_2 (y-x) + \frac{1}{36} (x+y)^3$$

- $(D^3 - 3DD'^2 - 2D'^3) z = \cos (x+2y)$

$m^3 - 3m - 2,$

$\Rightarrow (D^3 - 3DD'^2 - 2D'^3) z = \cos (x+2y)$

$8 \quad 2', 3 = r,$

$(m-2)$

$\Rightarrow (D+D')^2 \underline{(D-2D')} z = \cos (x+2y)$

$\therefore$ CF $= \phi_1 (x-y) + x \, \phi_2 (\cancel{2x} \; x-y) + \phi_3 (-2x-y)$

$= \phi_1 (y-x) + x . \phi_2 (y-x) + \phi_3 (y+2x)$, where ϕ_1, ϕ_2, ϕ_3 are arbitary functions

$\therefore$ P I $= \dfrac{1}{D^3 - 3DD'^2 - 2D'^3} \cos (x+2y)$

$= \dfrac{1}{1^3 - 3.1.4 - 2.8} \iiint \cos v \, dv \, dv \, dv$ where $v = x+2y$

$= - \dfrac{1}{27} \iint \sin v \, dv \, dv$

$= \dfrac{1}{27} \int \cos v \, dv = \dfrac{1}{27} \sin v = \dfrac{1}{27} \sin (x+2y)$

$\therefore$ The general solution is - -

$$Z = \phi_1 (y-x) + x \, \phi_2 (y-x) + \phi_3 (y+2x) + \frac{1}{27} \sin (x+2y)$$

• $(D^3 - 2D^2D' - DD'^2 + 2D'^3) = e^{x+y}$

$\Rightarrow (D^3 - 2D^2D' - DD'^2 + 2D'^3) = e^{x+y}$

$\Rightarrow (D-D')(D+D')(D-2D')z = e^{x+y}$

$\therefore$ CF $= \phi_1(-x-y) + \phi_2(x-y) + \phi_3(-2x-y)$

$= \phi_1(y+x) + \phi_2(y-x) + \phi_3(y+2x)$.

where ϕ_1, ϕ_2, ϕ_3 are arbitary functions

$\therefore$ P.I. $= \dfrac{1}{D^3 - 2D^2D' - DD'^2 + 2D'^3} e^{x+y}$

$= \dfrac{1}{(D-D')} \cdot \left\{ \dfrac{1}{D^2 - DD' - 2D'^2} e^{x+y} \right\}$

$= \dfrac{1}{D-D'} \cdot \dfrac{1}{1^2 - 1 - 2.1^2} \iint e^v\, dv\, dv$ where $v = x+y$

$= -\dfrac{1}{2} \cdot \dfrac{1}{D-D'} e^{x+y} = \left(-\dfrac{1}{2}\right) \dfrac{x}{1.1!} e^{x+y} = -\dfrac{x}{2} e^{x+y}$

[II]:- If $F(D,D')$ be homogeneous function of D and D' of degree n, then

$\cancel{\dfrac{1}{F(D,D')} \phi^n(ax+by) =}$ $\dfrac{1}{(bD-aD')^n} \phi(ax+by) = \dfrac{x^n}{b^n . n!} \phi(ax+by)$]

$\therefore$ The general solution is.

$z = \phi_1(y+x) + \phi_2(y-x) + \phi_3(y+2x) - \dfrac{x}{2} e^{x+y}$

• $(D^2 - 6DD' + 9D'^2)z = 12x^2 + 36xy$

$\Rightarrow (D^2 - 6DD' + 9D'^2)z = 12x^2 + 36xy$

$\Rightarrow (D-3D')^2 z = 12x^2 + 36xy$

$\therefore$ CF $= \phi_1(-3x-y) + x\,\phi_2(-3x-y)$

$= \phi_1(y+3x) + x\,\phi_2(y+3x)$

$D^3 - 7DD'^2 - 6D'^3$

$m^3 - 7m - 6 = 0$

$-8 + 14$

$-8 + 14$

$$\therefore \ P.I. = \frac{1}{(D-3D')^2}(12x^2+36xy)$$

$$= 12 \cdot \frac{1}{D^2\left(1-3D'/D\right)^2}(x^2+3xy)$$

$$= \frac{12}{D^2}\left(1-\frac{3D'}{D}\right)^{-2}(x^2+3xy)$$

$$= \frac{12}{D^2}\left(1+\frac{6D'}{D}+\cdots\right)(x^2+3xy)$$

$$= \frac{12}{D^2}\left\{x^2+3xy+\frac{6}{D}(3x)\right\}$$

$$= \frac{12}{D^2}\left(x^2+3xy+18 \cdot \frac{x^2}{2}\right)$$

$$= \frac{12}{D} \cdot \frac{1}{D}\left(10x^2+3xy\right)$$

$$= \frac{12}{D} \cdot \left(10\frac{x^3}{3}+3y \cdot \frac{x^2}{2}\right)$$

$$= \cancel{12}\ 12\left(\frac{10}{3} \cdot \frac{x^4}{4}+\frac{3y}{2} \cdot \frac{x^3}{3}\right)$$

$$= 10x^4+6x^3y$$

$\therefore$ The general solution is --

$$z = \phi_1(y+3x)+x\,\phi_2(y+3x)+10x^4+6x^3y$$

- $(D^2+DD'-6D'^2)\,z = y\cos x$ 16 2

$$\Rightarrow (D^2+DD'-6D'^2)\,z = y\cos x$$

$$\Rightarrow (D-2D')(D+3D')\,z = y\cos x$$

$$\therefore \ C.F = \phi_1(-2x-y)+\phi_2(3x-y)$$

$$= \phi_1(y+2x)+\phi_2(y-3x)$$

where ϕ_1, ϕ_2 are arbitary functions

$$\therefore \text{P.I.} = \frac{1}{(D-2D')(D+3D')} y\cos x$$

$$= \frac{1}{(D-2D')} \frac{1}{D}\left(1+\frac{3D'}{D}\right)^{-1} y\cos x$$

$$= \frac{1}{(D-2D')} \frac{1}{D}\left(1-\frac{3D'}{D}+\cdots\right) y\cos x$$

$$= \frac{1}{D-2D'} \frac{1}{D}(y\cos x - 3\sin x)$$

$$= \frac{1}{D-2D'} (y\sin x + 3\cos x)$$

$$= \frac{1}{D}\left(1-\frac{2D'}{D}\right)^{-1} (y\sin x + 3\cos x)$$

$$= \frac{1}{D}\left(1+\frac{2D'}{D}+\cdots\right) (y\sin x + 3\cos x)$$

$$= \frac{1}{D}\left(y\sin x + 3\cos x + 2(-\cos x)\right)$$

$$= \frac{1}{D}(y\sin x + \cos x)$$

$$= -y\cos x + \sin x$$

$$= \sin x - y\cos x$$

Alternative :-

$$\text{P.I} = \frac{1}{(D-2D')(D+3D')} y\cos x$$

Let $\quad u = \frac{1}{D+3D'} y\cos x$

$$\frac{\partial u}{\partial x} + 3\frac{\partial u}{\partial y} = y\cos x$$

$\therefore$ Lagrange's A.E. are . .

$$\frac{dx}{1} = \frac{dy}{3} = \frac{du}{y\cos x}$$

$$\therefore \frac{dx}{1} = \frac{dy}{3}$$

$$\Rightarrow y - 3x = c$$

$$\Rightarrow y = c + 3x$$

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$\therefore$ ~~P.I. = $\frac{1}{D-2D'}$~~

Now, taking 1st and 3rd fraction we get.

$$\frac{dx}{1} = \frac{\cancel{dy}\,du}{y\cos x}$$

$$du = (c+3x)\,\cancel{\cos dx}\,\cos x\,dx$$

$$u = \int (c+3x)\cos x\,dx$$

$$\therefore \quad P.I = \frac{1}{D-2D'} \int (c+3x)\cos x\,dx$$

$$= \cancel{\frac{1}{D-2D'} \int y \cos x}\,dx$$

$$= \frac{1}{D-2D'} \left[(3x+c)\sin x - \int 3 \sin x\,dx \right]$$

$$= \frac{1}{D-2D'} (y \sin x + 3\cos x)$$

Let $v = \frac{1}{D-2D'} (y \sin x + 3\cos x)$

$$\Rightarrow \quad \frac{\partial v}{\partial x} - 2\frac{\partial v}{\partial y} = y \sin x + 3\cos x$$

$\therefore$ Lagrange's A.E. are..

$$\frac{dx}{1} = \frac{dy}{-2} = \frac{\cancel{dz}\,dv}{y \sin x + 3\cos x}$$

$$\therefore \quad dx = -\frac{dy}{2} \Rightarrow y + 2x = c_1$$

$\therefore$ taking ~~of~~ 1st and last fractions we get.

$$dx = \frac{dv}{y \sin x + 3\cos x}$$

$$dv = \{(c_1 - 2x)\sin x + 3\cos x\}\,dx$$

$$v = \int (c_1 - 2x)\sin x + 3\cos x)\,dx$$

$\Rightarrow v = (c_1 - 2x)(-\cos x) - \int(-2)(-\cos x)\,dx + 3\sin x$

$\Rightarrow v = -y\cos x - 2\sin x + 3\sin x$

$\Rightarrow v = \sin x - y\cos x$

$\therefore$ The general solution is..

$$z = \phi_1(y+2x) + \phi_2(y-3x) + \sin x - y\cos x$$

Non-homogeneous PDE :-

$$F(D,D')z = f(x,y)$$

The method of finding a particular integral of non-homogeneous PDE are very similar to those of ODE with constant co-efficients.

Case-I :- Let $f(x,y) = e^{ax+by}$

$$\therefore P.I. = \frac{1}{F(D,D')} e^{ax+by}$$

$$= \frac{1}{F(a,b)} e^{ax+by} \quad , F(a,b) \neq 0$$

case-II :- If $f(x,y) = \sin(ax+by)$ or $\cos(ax+by)$

$$\therefore P.I. = \frac{1}{F(D,D')} f(x,y) = \frac{1}{F(D,D')} \sin(ax+by)$$

put $D^2 = -a^2,\ D'^2 = -b^2,\ DD' = -ab$

case-III :- If $f(x,y) = x^m y^n$

$$P.I. = \frac{1}{F(D,D')} f(x,y) = [F(D,D')]^{-1} x^m y^n$$

case-IV :- If $f(x,y) = V e^{ax+by}$

$$\therefore P.I. = \frac{1}{F(D,D')} f(x,y) = \frac{1}{F(D,D')} V e^{ax+by}$$

$$= e^{ax+by} \frac{1}{F(D+a, D'+b)} V$$

- $(DD'+D-D'-1)z = x+y$

$\Rightarrow (DD'+D-D'-1)z = x+y$

$\Rightarrow (D-1)(D'+1)z = x+y$

$\therefore$ C.F $= e^{x}\phi_1(-y) + e^{-y}\phi_2(x)$

$= e^{x}\phi_1(y) + e^{-y}\phi_2(x)$

$\therefore$ P.I $= \dfrac{1}{(D-1)(D'+1)}\, x+y$

$= \dfrac{1}{D-1}(1+D')^{-1}(x+y)$

$= \dfrac{1}{D-1}(1-D'+D'^2---)(x+y)$

$= \dfrac{1}{D-1}(x+y-1)$

$= -\dfrac{1}{1-D}(x+y-1)$

$= -(1-D)^{-1}(x+y-1)$

$= -(1+D+D^2+---)(x+y-1)$

$= -(x+y-1+1)$

$= -(x+y)$

$\therefore$ The general solution is ...

$$z = e^{x}\phi_1(y) + e^{-y}\phi_2(x) - (x+y)$$

- $(DD'+D-D'-1)z = xy$

$\Rightarrow (DD'+D-D'-1)z = xy$

$\Rightarrow (D-1)(D'+1)z = xy$

$\therefore$ C.F $= e^{x}\phi_1(-y) + e^{-y}\phi_2(x)$

$= e^{x}\phi_1(y) + e^{-y}\phi_2(x)$

$$P.I. = \frac{1}{(D-1)(D'+1)} xy$$

$$= \frac{1}{D-1} (1+D')^{-1} xy$$

$$= \frac{1}{D-1} (1-D'+D'^2 - \cdots) xy$$

$$= \frac{1}{D-1} (xy - x)$$

$$= -(1-D)^{-1} (xy-x)$$

$$= -(1+D+D^2+\cdots)(xy-x)$$

$$= -(xy - x + y - 1)$$

$$= -(x+1)(y-1)$$

$\therefore$ The general solution is ....

$$z = e^{x} \phi_1(y) + e^{-y} \phi_2(x) - (x+1)(y-1)$$

- $(D^2 - DD' + D' - 1) z = \cos(x+2y) + e^{y}$

$\Rightarrow (D^2 - DD' + D' - 1) z = \cos(x+2y) + e^{y}$

$\Rightarrow (D-1)(D-D'+1) z = \cos(x+2y) + e^{y}$

$\therefore C.F. = e^{x} \phi_1(-y) + e^{-x} \phi_2(-x-y)$

$$= e^{x} \phi_1(y) + e^{-x} \phi_2(y+x)$$

$\therefore P.I. = \frac{1}{(D-1)(D-D'+1)} [\cos(x+2y) + e^{y}]$

$$= \frac{1}{D^2 - DD' + D' - 1} \cdot \cos(x+2y) + \frac{1}{(D-D'+1)} \cdot \frac{1}{D-1} e^{y} \quad ??$$

$$= \frac{1}{-1+2+D'-1} \cos(x+2y) + \frac{1}{(D-D'+1)} (-e^{y})$$

$$= \frac{1}{D'} \cos(x+2y) + -\frac{1}{D-D'+1} e^{y}$$

$$= \frac{D'}{D'^2}\cos(x+2y) - e^y \frac{1}{D-D'}\cdot 1$$

$$= -\frac{1}{4}\cdot D'\cos(x+2y) - e^y \frac{1}{D(1-D'/D)}\cdot 1$$

$$= -\frac{1}{4}\left[-\sin(x+2y).2\right] - e^y\cdot \frac{1}{D}\cdot 1$$

$$= \frac{1}{2}\sin(x+2y) - xe^y$$

∴ The general solution is - -

$$z = e^x \phi_1(y) + e^{-x}\phi_2(y+x) + \frac{1}{2}\sin(x+2y) - xe^y$$

- ~~$(D^2-D'^2-3D$~~ $(D-1)(D-D'+1)z = \cos(x+2y) + e^y + xy + 1$

$\Rightarrow$ $(D-1)(D-D'+1) = \cos(x+2y) + e^y + xy + 1$

$C.F = \cancel{\phi}\ e^{x}\phi_1(-y) + e^{-x}\phi_2(-x-y)$

$$= e^x \phi_1(y) + e^{-x}\phi_2(y+x)$$

$$P.I. = \frac{1}{(D-1)(D-D'+1)}\left[\cos(x+2y) + e^y + xy + 1\right]$$

Now $\frac{1}{(D-1)(D-D'+1)}\cos(x+2y)$

$$= \frac{1}{D^2 - DD' + D' - 1}\cos(x+2y)$$

$$= \frac{1}{-1+2+D'-1}\cos(x+2y)$$

$$= \frac{1}{D'}\cos(x+2y)$$

$$= \frac{D'}{D'^2}\cos(x+2y)$$

$$= -\frac{1}{4}\cdot D'(\cos(x+2y))$$

$$= -\frac{1}{4}\left(-\sin(x+2y).2\right) = \frac{1}{2}\sin(x+2y)$$

$$= \frac{1}{(D-1)(D-D'+1)} e^{y}$$

$$= \frac{1}{D-D'+1} \cdot \frac{1}{D-1} e^{y}$$

$$= \frac{1}{D-D'+1} (-e^{y})$$

$$= -e^{y} \frac{1}{D-(D'+1)+1} 1$$

$$= -e^{y} \frac{1}{D-D'} 1$$

$$= -e^{y} \frac{1}{D(1-D'/D)} 1$$

$$= -e^{y} \frac{1}{D} \left(1+ \frac{D'}{D} + \cdots\right) 1$$

$$= -e^{y} \frac{1}{D} 1$$

$$= -xe^{y}$$

$$\therefore \frac{1}{(D-1)(D-D'+1)} (xy+1)$$

$$= \frac{1}{(D-1)} (1+D-D')^{-1} (xy+1)$$

$$= \frac{1}{(D-1)} \left(1-(D-D')+(D-D')^2 - \cdots\right) (xy+1)$$

$$= \frac{1}{(D-1)} \left(1-D+D'+D^2-2DD'+D'^2 - \cdots\right) (xy+1)$$

$$= \frac{1}{(D-1)} (xy+1-y+x-2)$$

$$= -(1-D)^{-1} (xy-y+x-1)$$

$$= -(1+D+D^2+\cdots)(xy-y+x-1)$$

$$= -(xy-y+x-1+y+1)$$

$$= -(xy+x)$$

$\therefore$ P.I. $= \frac{1}{2}\sin(x+2y) - xe^{y} - (xy+x)$

$\therefore$ The general solution is . .

$$Z = e^{x}\phi_1(y) + e^{-x}\phi_2(y+x) + \frac{1}{2}\sin(x+2y) - xe^{y} - (xy+x)$$

- $(D^2 - D'^2 - 3D + 3D')z = xy + e^{x+2y}$

$\Rightarrow (D^2 - D'^2 - 3D + 3D')z = xy + e^{x+2y}$

$\Rightarrow (D-D')(D+D'-3) = xy + e^{x+2y}$

$\therefore$ C.F. $= \phi_1(-x-y) + e^{3x}\phi_2(x-y)$

$= \phi_1(y+x) + e^{3x}\phi_2(y-x)$

$\therefore$ P.I. $= \frac{1}{(D-D')(D+D'-3)}\{xy + e^{x+2y}\}$

Now, $\frac{1}{(D-D')(D+D'-3)}xy$

$= \frac{1}{D-D'}\left(-\frac{1}{3}\right)\left(1 \cancel{+} - \frac{D+D'}{3}\right)^{-1}xy$

$= \frac{1}{D-D'}\left(-\frac{1}{3}\right)\left(1 + \frac{D+D'}{3} + \left(\frac{D+D'}{3}\right)^2 + \cdots\right)xy$

$= -\frac{1}{3}\cdot\frac{1}{D-D'}\left(xy + \frac{y}{3} + \frac{x}{3} + \frac{1}{9}2\right)$

$= -\frac{1}{3}\frac{1}{D}\left(1 - \frac{D'}{D}\right)^{-1}\left(xy + \frac{y}{3} + \frac{x}{3} + \frac{2}{9}\right)$

$= -\frac{1}{3}\frac{1}{D}\left(1 + \frac{D'}{D} + \frac{D'^2}{D^2} + \cdots\right)\left(xy + \frac{y}{3} + \frac{x}{3} + \frac{2}{9}\right)$

$= -\frac{1}{3}\cdot\frac{1}{D}\left(xy + \frac{y}{3} + \frac{x}{3} + \frac{2}{9} + \frac{x^2}{2} + \frac{x}{3}\right)$

$= -\frac{1}{3}\frac{1}{D}\left(xy + \frac{y}{3} + \frac{2x}{3} + \frac{x^2}{2} + \frac{2}{9}\right)$

$= -\frac{1}{3}\left(x^2y/2 + \frac{xy}{3} + \frac{x^2}{3} + \frac{x^3}{6} + \frac{2}{9}x\right)$

$= -\frac{x^2y}{6} - \frac{xy}{9} - \frac{x^2}{9} - \frac{x^3}{18} - \frac{2}{27}x$

$\therefore \frac{1}{(D-D')(D+D'-3)}e^{x+2y}$

$= \frac{1}{D+D'-3}\cdot\frac{1}{D-D'}e^{x+2y}$

$= \frac{1}{D+D'-3}\frac{1}{1-2}e^{x+2y}$

$= -\frac{1}{D+D'-3}e^{x+2y}$

$= -\cancel{1}$

$= -e^{x+2y}\frac{1}{(D+1)+(D'+2)-3}\cdot 1$

$= -e^{x+2y}\frac{1}{D+D'}1$

$= -e^{x+2y}\frac{1}{D}\left(1+\frac{D'}{D}\right)^{-1}1$

$= -e^{x+2y}\frac{1}{D}\left(1-\frac{D'}{D}+\cdots\right)1$

$= -e^{x+2y}\cdot\frac{1}{D}1$

$= -xe^{x+2y}$

$\therefore$ P.I $= -\frac{1}{6}x^2y - \frac{1}{9}xy - \frac{1}{9}x^2 - \frac{1}{18}x^3$
$- \frac{2}{27}x - xe^{x+2y}$

$\therefore$ The general solution is :-

$$z = \phi_1(y+x) + e^{3x}\phi_2(y-x) - \frac{1}{6}x^2y - \frac{1}{9}xy - \frac{1}{9}x^2 - \frac{1}{18}x^3 - \frac{2}{27}x - xe^{x+2y}$$

- $(D^2 - D')z = e^{x+y} + 5\cos(x+2y)$

$\Rightarrow$ Since $(D^2 - D')$ can't be resolved into linear factors in D and D'

Let $z = Ae^{hx+ky}$ be a trial solution of $(D^2 - D)z = 0$.

$\therefore \quad D'z = Ae^{hx+ky}\,k$

$D^2z = Ah^2e^{hx+ky}$

$\therefore \quad Ae^{hx+ky}(h^2 - k) = 0$

$\Rightarrow \quad k = h^2$

$\therefore \quad C.F = \sum Ae^{hx+ky} = \sum Ae^{hx+h^2y}$, A, h being arbitary constants

$\therefore$ P.I. $= \dfrac{1}{D^2 - D'}\left[e^{x+y} + 5\cos(x+2y)\right]$

$$= \frac{1}{D^2 - D'}e^{x+y} + 5\frac{1}{D^2 - D'}\cos(x+2y)$$

$$= \cancel{e^{x+y}\frac{1}{D'}}\; e^{x+y}\frac{1}{D^2 + 2D - D'}1 + 5.\frac{1}{-1 - D'}\cancel{\cos}\cos(x+2y)$$

$$= -e^{x+y}\frac{1}{D'}\left(1 - \frac{D^2 + 2D}{D'}\right)^{-1}.1 - 5\frac{1}{D'+1}\cos(x+2y)$$

$$= -e^{x+y}\frac{1}{D'}\left(1 + \frac{D^2 + 2D}{D'} + \cdots\right)1 - 5\frac{D'\cancel{+1}-1}{D'^2 - 1}\cos(x+2y)$$

$$= -ye^{x+y} - 5.\frac{1}{(-5)}\left(-\sin(x+2y)\,2 - \cos(x+2y)\right)$$

$$= -ye^{x+y} - 2\sin(x+2y) - \cos(x+2y)$$

$\therefore$ The general solution is ....

$$z = \sum Ae^{hx+h^2y} - ye^{x+y} - 2\sin(x+2y) - \cos(x+2y)$$

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- Problem based on PDE of Euler - Cauchy type :-

- $(x^2D^2 + 2xy\,DD' + y^2D'^2)z = x^2y^2$

$\Rightarrow$ Let $x = e^u \qquad y = e^v$

$u = \log x \qquad v = \log y$

$\therefore Dz = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} = \frac{1}{x}\frac{\partial z}{\partial u}$

$\Rightarrow (xD)z = \frac{\partial z}{\partial u} = D_1 z \quad (\because \frac{\partial}{\partial u} \equiv D_1)$

$\therefore D(xD)z = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) = \frac{\partial}{\partial u}\cdot\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x}$

$= \frac{\partial^2 z}{\partial u^2}\cdot\frac{1}{x}$

$\Rightarrow Dz + xD^2z = \frac{1}{x}D_1^2 z$

$\Rightarrow (x^2D^2)z = D_1^2z - xDz = D_1^2z - D_1z = D_1(D_1-1)z$

Similarly $(yD')z = D_1'z \quad (\frac{\partial}{\partial v} \equiv D_1')$

and $(y^2D'^2)z = D_1'(D_1'-1)z$

$\therefore DD'z = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}z\right)$

$= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}\right)$

$= \frac{\partial}{\partial x}\left(\frac{1}{y}\frac{\partial z}{\partial v}\right) = \frac{1}{y}\cdot\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\cdot\frac{\partial u}{\partial x} = \frac{1}{xy}D_1D_1'z$

$\therefore xy\,DD'z = D_1D_1'z$

$\therefore (x^2D^2 + 2xy\,DD' + y^2D'^2)z = x^2y^2$

$\Rightarrow \{D_1^2 - D_1 + 2D_1D_1' + D_1'^2 - D_1'\}z = e^{2u+2v}$

$\Rightarrow (D_1+D_1')(D_1+D_1'-1) = e^{2u+2v}$

$\therefore C.F = \phi_1(v-u) + e^u\phi_2(v-u)$

$= \phi_1(\log y/x) + x\,\phi_2(\log y/x)$

$\therefore$ P. I $= \dfrac{1}{(D_1+D_1')(D_1+D_1'-1)} e^{2u+2v}$

$= \dfrac{1}{D_1+D_1'} \cdot \dfrac{1}{2+2-1} e^{2u+2v}$

$= \dfrac{1}{3} \dfrac{1}{D_1+D_1'} e^{2u+2v}$

$= \dfrac{1}{3} \dfrac{1}{2+2} e^{2u+2v}$

$= \dfrac{1}{12} x^2y^2$

$\therefore$ The general solution is ... $z = \phi_1(\log y/x) + x\,\phi_2(\log y/x) + \frac{1}{12}x^2y^2$

- $(x^2D^2 - 4xy\,DD' + 4y^2D'^2 + 4yD' + xD)z = x^2y$

$\Rightarrow$ Let $x = e^u$ $\qquad y = e^v$

$u = \log x \qquad v = \log y$

$\therefore Dz = \dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial u} \cdot \dfrac{\partial u}{\partial x} = \dfrac{1}{x} D_1 z \quad \left(\dfrac{\partial}{\partial u} \equiv D_1\right)$

$(xD)z = D_1 z$

$D(xD)z = \dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial u}\right) = \dfrac{\partial}{\partial u}\left(\dfrac{\partial z}{\partial u}\right) \cdot \dfrac{\partial u}{\partial x}$

$\Rightarrow Dz + x\,D^2z = \dfrac{1}{x} D_1^2 z$

$\Rightarrow (x^2D^2)z = D_1^2z - xDz = D_1^2z - D_1z = D_1(D_1-1)z$

similarly $(yD')z = D_1'z \quad \left(\dfrac{\partial}{\partial v} \equiv D_1'\right)$

and $(y^2D'^2)z = D_1'(D_1'-1)z$

$\therefore DD'z = \dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial y}\right)$

$= \dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial v} \dfrac{\partial v}{\partial y}\right)$

$= \dfrac{\partial}{\partial x}\left(\dfrac{1}{y} \dfrac{\partial z}{\partial v}\right)$

$= \dfrac{\partial}{\partial u}\left(\dfrac{1}{y} \dfrac{\partial z}{\partial v}\right) \cdot \dfrac{\partial u}{\partial x}$

$= \dfrac{1}{xy} \cdot D_1D_1'z$

$\therefore xy\,DD'z = D_1D_1'z$

$\Rightarrow \quad Dz + \cancel{(x^2D^2)z} \; xD^2 z = \frac{1}{x} D_1^2 z$

$\Rightarrow \quad (x^2D^2)z = (D_1^2 - D_1)z$

similarly $\quad (yD')z = D_1'z \qquad \left(\frac{\partial}{\partial v} \equiv D_1'\right)$

$\therefore \quad (y^2D'^2)z = (D_1'^2 - D_1')z$

$\therefore \quad DD'z = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}\right)\cdot\frac{\partial u}{\partial x}$

$= \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\cdot\frac{1}{y}\right)\cdot\frac{1}{x}$

$xy\,DD'z = D_1D_1'z$

$\therefore \quad (x^2D^2 - 2xy\,DD' - 3y^2D'^2 + xD - 3yD')z = x^2y\cos(\log x^2)$

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$\Rightarrow \quad (D_1^2 - D_1 - 2D_1D_1' - 3D_1'^2 + 3D_1 + D_1 - 3D_1')z = e^{2u+v}\cos 2u$

$\Rightarrow \quad (D_1^2 - 2D_1D_1' - 3D_1'^2) = e^{2u+v}\cos 2u$

$\Rightarrow \quad (D_1 + D_1')(D_1 - 3D_1') = e^{2u+v}\cos 2u$

$\therefore \quad C.F. = \phi_1(u - v) + \phi_2(-3u - v)$

$= \phi_1(v - u) + \phi_2(v + 3u)$

$= \phi_1(\log y/x) + \phi_2(\log x^3y)$

$= \psi_1(y/x) + \psi_2(x^3y)$

$\therefore \quad P.I. = \frac{1}{(D_1 + D_1')(D_1 - 3D_1')} e^{2u+v}\cos 2u$

$= e^{2u+v}\frac{1}{(D_1 + 2 + D_1' + 1)(D_1 + 2 - 3D_1' - 3)}\cos 2u$

$= e^{2u+v}\frac{1}{(D_1 + D_1' + 3)(D_1 - 3D_1' - 1)}\cos 2u$

$= e^{2u+v}\frac{1}{D_1^2 - 2D_1D_1' - 3D_1'^2 + 2D_1 - 10D_1' - 3}\cos 2u$

$= e^{2u+v}\frac{1}{-4 + 2D_1 - 10D_1' - 3}\cos 2u$

$$= e^{2u+v} \frac{1}{(2D_1 - 10D_1') - 7} \cos 2u$$

$$= e^{2u+v} \frac{2D_1 - 10D_1' + 7}{(2D_1 - 10D_1')^2 - 49} \cos 2u$$

$$= e^{2u+v} \frac{2D_1 - 10D_1' + 7}{4D_1^2 - 40D_1D_1' + 100D_1'^2 - 49} \cos 2u$$

$$= e^{2u+v} \frac{2D_1 - 10D_1' + 7}{-16 - 49} \cos 2u$$

$$= -\frac{1}{65} e^{2u+v} (-4 \sin 2u + 7 \cos 2u)$$

$$= \frac{1}{65} x^2 y \left\{ 4 \sin (\log x^2) - 7 \cos (\log x^2) \right\}$$

$\therefore$ The general solution is --

$$z = \varphi_1 (y/x) + \varphi_2 (x^3 y) + \frac{1}{65} x^2 y \left\{ 4 \sin (\log x^2) - 7 \cos (\log x^2) \right\}$$

- **2nd order PDE :-**

A PDE is said to be 2nd order semilinear PDE if it can be written as..

$$R(x,y)\, u_{xx} + S(x,y)\, u_{xy} + T(x,y)\, u_{yy} + g(x,y,u,u_x,u_y) = 0 \quad \text{--- (i)}$$

where R, S, T are continuous functions of x and y s.t.

$$R^2 + S^2 + T^2 \neq 0$$

A function $u = u(x,y)$ is said to be a regular solution of equation (i) in a domain $D \subseteq R \times R$ if $u \in C^2(D)$ and the function and its derivatives satisfy the equation (i) identically in x and y for $(x,y) \in D$

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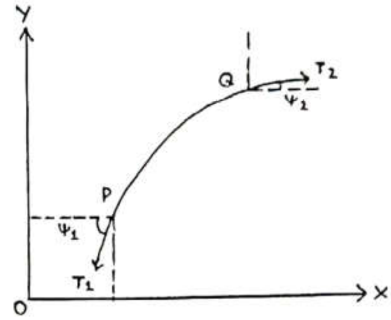
- **Transverse vibration of a string :-**

Let $y = y(t)$ be a transverse displacement from the mean position (x-axis) of a string at time t at the point x.

Consider a small position of the string Δs between the tensions, as at P and Q are T_1 and T_2 respectively and tensions making angles ψ_1 and ψ_2 with x-axis respectively.



Now, we ~~neglect~~ neglect the weight of the string. The equation of motion are....

(i) in x-direction (assuming no-displacement in x-direction)

$$T_2 \cos\psi_2 = T_1 \cos\psi_1 = T \text{ (say)}$$

(ii) in y-direction

$$(\rho \Delta s)\, y_{tt} = T_2 \sin\psi_2 - T_1 \sin\psi_1$$ (ρ = linear density of the string)

$$= T(\tan\psi_2 - \tan\psi_1) \text{ .. using (i)}$$

$$\tan\psi_1 = (y_x)|_P$$

$$\tan\psi_2 = (y_x)|_Q$$

$$\simeq (y_x)|_P + (y_{xx})|_P \Delta x$$

$$\therefore \quad (\rho\Delta s)\, y_{tt} = T\{(y_x)|_P + (y_{xx})|_P \Delta x - (y_x)|_P\}$$

$$= T(y_{xx})|_P \Delta x$$

$$\Rightarrow \rho \frac{\Delta s}{\Delta x} y_{tt} = T(y_{xx})|_P$$

taking limit as $\Delta s \to 0$ and $\Delta x \to 0$, we get

$$\rho \frac{ds}{dx} y_{tt} = T y_{xx}$$

Now, $ds^2 = dx^2 + dy^2$

$$\frac{ds}{dx} = \sqrt{1 + y_x^2}$$

$$\rho y_{tt} = \frac{T}{\sqrt{1+y_x^2}} y_{xx}$$

if $|y_x| << 1$, then

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$$\rho y_{tt} = T y_{xx}$$

$$y_{xx} = \frac{\rho}{T} y_{tt}$$

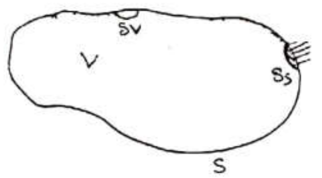
$$y_{xx} = \frac{1}{c^2} y_{tt} \qquad [\because c^2 = T/\rho]$$

$\therefore$ which is the required one-dimensional wave equation.

- **Heat Equation :-**

Let us consider a homogeneous, isotrophic (homogeneous means that the material properties are translation invariant and isotrophic means the material properties are same in all directions.) solid.

Let V be any arbitary volume inside the solid bounded by a surface S. Let δV be a volume element. The heat energy store in δV is equal to $c\rho u \delta V$



where c is the specific heat of solid ρ is the density and u is the temperature which is a function of position and time.

$\therefore$ Total Heat energy in $V = \iiint_V c\rho u \, dV$

Let δs be a surface element. The heat flow across $ds = k \nabla u . \bar{n} \, ds$ where $\bar{n}$ is outward drawn normal to the surface S and k is the thermal conductivity of the solid.

$\therefore$ Total Heat flux across $S = \iint_S k \nabla u \, \bar{n} \, ds$

Now using divergence theorem --

$$\iint_S k \nabla u \, \bar{n} \, ds = \iiint_V \nabla (k \nabla u) \, dv$$

$\therefore$ The rate of change of heat energy in V = the heat flux across S

$$\frac{\partial}{\partial t} \iiint_V c\rho u \, dv = \iiint_V \nabla (k \nabla u) \, dv$$

$$\Rightarrow \iiint_V \left\{ \frac{\partial}{\partial t} (c\rho u) - \nabla (k \nabla u) \right\} dv = 0$$

as V is arbitrary, to satisfy this equation, the co-efficient of dv must be zero.

$$\therefore \frac{\partial}{\partial t}(c\rho u) - \nabla(k\nabla u) = 0$$

$$c\rho \frac{\partial u}{\partial t} - \nabla(k\nabla u) = 0$$

if the thermal conductivity k is constant, then..

$$c\rho \frac{\partial u}{\partial t} - k\nabla^2 u = 0$$

$$\Rightarrow \frac{\partial u}{\partial t} = \frac{k}{c\rho}\nabla^2 u$$

$$\Rightarrow \frac{\partial u}{\partial t} = \eta \nabla^2 u \qquad [\eta = k/c\rho]$$

which is the required Heat equation.

In one dimensional case.. $\frac{\partial u}{\partial t} = \eta \frac{\partial^2 u}{\partial x^2}$

- **Classification of 2nd order PDE :-**

Consider $R(x,y)\,u_{xx} + S(x,y)\,u_{xy} + T(x,y)\,u_{yy} + g(x,y,u,u_x,u_y) = 0$

where R, S, T are continuous functions of x and y s.t.

$$R^2 + S^2 + T^2 \neq 0$$

or $Lu + g(x,y,u,u_x,u_y) = 0$

where $L = R\frac{\partial^2}{\partial x^2} + S\frac{\partial^2}{\partial x \partial y} + T\frac{\partial^2}{\partial y^2}$

If $S^2 - 4RT > 0$ (Hyperbolic)

$S^2 - 4RT = 0$ (Parabolic)

$S^2 - 4RT < 0$ (elliptic)

- **Cannonical transformation :-**

$$R(x,y)\,u_{xx} + S(x,y)\,u_{xy} + T(x,y)\,u_{yy} + g(x,y,u,u_x,u_y) = 0 \quad \text{...(i)}$$

(i) $S^2 - 4RT > 0$ (hyperbolic) e.g. wave equation

(ii) $S^2 - 4RT = 0$ (Parabolic) e.g. Heat equation

(iv) $S^2 - 4RT < 0$ (elliptic) e.g. Laplace equation